ECE 331 Lab #4
Ampere's Force Experiment
Derivation

Introduction

Ampere's law of force describes the interaction between two current-carrying circuits. This mathematical development will begin with fundamental equations of magneto-statics, derive Ampere's law of force, and then apply it to a real problem to calculate the equilibrium separation between two current-carrying loops of wire.

Mathematical Development

Start with the Lorentz force and the Law of Bio-Savart, assuming no significant electric field and a quasi-static magnetic field.

$$\boldsymbol{F} = q(\boldsymbol{v} \times \boldsymbol{B}) \tag{1}$$

$$\boldsymbol{B} = \frac{\mu_\circ}{4\pi} \oint_C \frac{I d\boldsymbol{l} \times \boldsymbol{a}_R}{R^2} \tag{2}$$

Apply equations 1 and 2 to the problem geometry shown in class by taking the differential of the Lorentz force and substituting in the appropriate subscripts:

$$d\boldsymbol{F}_{21} = I_2 d\boldsymbol{l_2} \times \boldsymbol{B_1} \tag{3}$$

$$\boldsymbol{B}_1 = \frac{\mu_\circ}{4\pi} \oint_{l_1} \frac{I_1 d\boldsymbol{l}_1 \times \boldsymbol{a}_{R_{21}}}{R_{21}^2} \tag{4}$$

The force $\boldsymbol{F}_{21}$ may be calculated directly by substituting equation 4 into equation 3 and integrating. Note that $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = -\boldsymbol{A} \times (\boldsymbol{C} \times \boldsymbol{B}) = +(\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A}$.

$$\boldsymbol{F}_{21} = \oint_{l_2} d\boldsymbol{F}_{21} \tag{5}$$

$$= \oint_{l_2} I_2 d\boldsymbol{l}_2 \times \frac{\mu_\circ}{4\pi} \oint_{l_1} \frac{I_1 d\boldsymbol{l}_1 \times \boldsymbol{a}_{R_{21}}}{R_{21}^2} \tag{6}$$

$$= \frac{\mu_\circ I_1 I_2}{4\pi} \oint_{l_2} \oint_{l_1} \frac{d\boldsymbol{l}_2 \times (d\boldsymbol{l}_1 \times \boldsymbol{a}_{R_{21}})}{R_{21}^2} \tag{7}$$

$$\boldsymbol{F}_{21} = \frac{\mu_\circ I_1 I_2}{4\pi} \oint_{l_2} \oint_{l_1} \frac{(\boldsymbol{a}_{R_{21}} \times d\boldsymbol{l}_1) \times d\boldsymbol{l}_2}{R_{21}^2} \tag{8}$$

Using the definitions $\boldsymbol{R}_{21} = \boldsymbol{R}_2 - \boldsymbol{R}_1$ and $\boldsymbol{a}_{R_{21}} = \frac{\boldsymbol{R}_{21}}{R_{21}}$, the force is:

$$\boldsymbol{F}_{21} = \frac{\mu_\circ I_1 I_2}{4\pi} \oint_{l_2} \oint_{l_1} \frac{((\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1) \times d\boldsymbol{l}_2}{|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3} \tag{9}$$

Where:

$$\boldsymbol{R}_1 = b_1 \boldsymbol{a}_r \tag{10}$$

$$\boldsymbol{R}_2 = b_2 \boldsymbol{a}_r + L\boldsymbol{a}_z \tag{11}$$

$$d\boldsymbol{l}_1 = b_1 \boldsymbol{a}_\phi d\phi_1 \tag{12}$$

$$d\boldsymbol{l}_2 = b_2 \boldsymbol{a}_\phi d\phi_2 \tag{13}$$

Before equation 9 can be integrated, the integrand must be written in Cartesian coordinates to allow summing.

$$\boldsymbol{R}_1 = b_1 (\cos\phi_1 \boldsymbol{a}_x + \sin\phi_1 \boldsymbol{a}_y) = b_1 \cos\phi_1 \boldsymbol{a}_x + b_1 \sin\phi_1 \boldsymbol{a}_y \tag{14}$$

$$\boldsymbol{R}_2 = b_2 (\cos\phi_2 \boldsymbol{a}_x + \sin\phi_2 \boldsymbol{a}_y) + L\boldsymbol{a}_z = b_2 \cos\phi_2 \boldsymbol{a}_x + b_2 \sin\phi_2 \boldsymbol{a}_y + L\boldsymbol{a}_z \tag{15}$$

$$d\boldsymbol{l}_1 = b_1 (-\sin\phi_1 \boldsymbol{a}_x + \cos\phi_1 \boldsymbol{a}_y) d\phi_1 = -b_1 \sin\phi_1 \boldsymbol{a}_x d\phi_1 + b_1 \cos\phi_1 \boldsymbol{a}_y d\phi_1 \tag{16}$$

$$d\boldsymbol{l}_2 = b_2 (-\sin\phi_2 \boldsymbol{a}_x + \cos\phi_2 \boldsymbol{a}_y) d\phi_2 = -b_2 \sin\phi_2 \boldsymbol{a}_x d\phi_2 + b_2 \cos\phi_2 \boldsymbol{a}_y d\phi_2 \tag{17}$$

Next compute the numerator of the integrand $((\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1) \times d\boldsymbol{l}_2$.

Step 1:

$$\begin{aligned} \boldsymbol{R}_2 - \boldsymbol{R}_1 &= b_2 \cos\phi_2 \boldsymbol{a}_x + b_2 \sin\phi_2 \boldsymbol{a}_y + L\boldsymbol{a}_z - b_1 \cos\phi_1 \boldsymbol{a}_x b_1 \sin\phi_1 \boldsymbol{a}_y & (18) \\ &= \boldsymbol{a}_x (b_2 \cos\phi_2 - b_1 \cos\phi_1)) + \boldsymbol{a}_y (b_2 \sin\phi_2 - b_1 \sin\phi_1) + \boldsymbol{a}_z (L) & (19) \end{aligned}$$

Step 2:

$$(\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1 = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ b_2 \cos\phi_2 - b_1 \cos\phi_1 & b_2 \sin\phi_2 - b_1 \sin\phi_1 & L \\ -b_1 \sin\phi_1 d\phi_1 & b_1 \cos\phi_1 d\phi_1 & 0 \end{vmatrix} \tag{20}$$

$$\begin{aligned} &= \boldsymbol{a}_x ((b_2 \sin\phi_2 - b_1 \sin\phi_1)(0) - (L)(b_1 \cos\phi_1 d\phi_1)) \\ &\quad -\boldsymbol{a}_y ((b_2 \cos\phi_2 - b_1 \cos\phi_1)(0) - (L)(-b_1 \sin\phi_1 d\phi_1)) \\ &\quad +\boldsymbol{a}_z \Big((b_2 \cos\phi_2 - b_1 \cos\phi_1)(b_1 \cos\phi_1 d\phi_1) \\ &\qquad - (b_2 \sin\phi_2 - b_1 \sin\phi_1)(-b_1 \sin\phi_1 d\phi_1) \Big) \end{aligned} \tag{21}$$

$$
\begin{aligned}
&= \boldsymbol{a}_x \left(-Lb_1 \cos\phi_1\right) d\phi_1 + \boldsymbol{a}_y \left(-Lb_1 \sin\phi_1\right) d\phi_1 \\
&\quad +\boldsymbol{a}_z \Big(b_1 b_2 \cos\phi_1 \cos\phi_2 - b_1^2 \cos^2\phi_1 \\
&\qquad\qquad + b_1 b_2 \sin\phi_1 \sin\phi_2 - b_1^2 \sin^2\phi_1 \Big) d\phi_1 \qquad (22) \\
&= -Lb_1 \cos\phi_1 \boldsymbol{a}_x d\phi_1 - Lb_1 \sin\phi_1 \boldsymbol{a}_y d\phi_1 \\
&\quad + \left(b_1 b_2 \left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right) - b_1^2 \left(\cos^2\phi_1 + \sin^2\phi_1\right) \right) \boldsymbol{a}_z d\phi_1 \qquad (23)
\end{aligned}
$$

$$
(\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1 = -Lb_1 \cos\phi_1 \boldsymbol{a}_x d\phi_1 - Lb_1 \sin\phi_1 \boldsymbol{a}_y d\phi_1 + \left(b_1 b_2 \cos\left(\phi_1 - \phi_2\right) - b_1^2 \right) \boldsymbol{a}_z d\phi_1 \qquad (24)
$$

Note that $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$.

Step 3:

$$
((\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1) \times d\boldsymbol{l}_2 = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ -Lb_1 \cos\phi_1 d\phi_1 & -Lb_1 \sin\phi_1 d\phi_1 & \left(b_1 b_2 \cos\left(\phi_1 - \phi_2\right) - b_1^2\right) d\phi_1 \\ -b_2 \sin\phi_2 d\phi_2 & b_2 \cos\phi_2 d\phi_2 & 0 \end{vmatrix} \qquad (25)
$$

$$
\begin{aligned}
&= \boldsymbol{a}_x \Big(\left(-Lb_1 \sin\phi_1 d\phi_1\right)(0) \\
&\qquad - \left(\left(b_1 b_2 \cos(\phi_1 - \phi_2) - b_1^2 \right) d\phi_1 \right) \left(b_2 \cos\phi_2 d\phi_2\right) \Big) \\
&\quad -\boldsymbol{a}_y \Big(\left(-Lb_1 \cos\phi_1 d\phi_1\right)(0) \\
&\qquad - \left(\left(b_1 b_2 \cos(\phi_1 - \phi_2) - b_1^2 \right) d\phi_1 \right) \left(-b_2 \sin\phi_2 d\phi_2\right) \Big) \\
&\quad +\boldsymbol{a}_z \Big(\left(-Lb_1 \cos\phi_1 d\phi_1\right) \left(b_2 \cos\phi_2 d\phi_2\right) \\
&\qquad - \left(-Lb_1 \sin\phi_1 d\phi_1\right) \left(-b_2 \sin\phi_2 d\phi_2\right) \Big) \qquad (26) \\
&= \boldsymbol{a}_x \left(-b_1 b_2^2 \cos(\phi_1 - \phi_2) \cos\phi_2 d\phi_1 d\phi_2 + b_1^2 b_2 \cos\phi_2 d\phi_1 d\phi_2 \right) \\
&\quad +\boldsymbol{a}_y \left(-b_1 b_2^2 \cos(\phi_1 - \phi_2) \sin\phi_2 d\phi_1 d\phi_2 + b_1^2 b_2 \sin\phi_2 d\phi_1 d\phi_2 \right) \\
&\quad +\boldsymbol{a}_z \Big(- Lb_1 b_2 \cos\phi_1 \cos\phi_2 d\phi_1 d\phi_2 - Lb_1 b_2 \sin\phi_1 \sin\phi_2 d\phi_1 d\phi_2 \Big) \qquad (27) \\
&= \boldsymbol{a}_x b_1 b_2 \cos\phi_2 \left(b_1 - b_2 \cos(\phi_1 - \phi_2)\right) d\phi_1 d\phi_2 \\
&\quad +\boldsymbol{a}_y b_1 b_2 \sin\phi_2 \left(b_1 - b_2 \cos(\phi_1 - \phi_2)\right) d\phi_1 d\phi_2 \\
&\quad -\boldsymbol{a}_z L b_1 b_2 \left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right) d\phi_1 d\phi_2 \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
((\boldsymbol{R}_2 - \boldsymbol{R}_1) \times d\boldsymbol{l}_1) \times d\boldsymbol{l}_2 &= b_1 b_2 \Big(\left(\boldsymbol{a}_x \cos\phi_2 + \boldsymbol{a}_y \sin\phi_2\right) \left(b_1 - b_2 \cos(\phi_1 - \phi_2)\right) \\
&\quad -\boldsymbol{a}_z L \cos(\phi_1 - \phi_2) \Big) d\phi_1 d\phi_2 \qquad (29)
\end{aligned}
$$

Finally, compute the denominator of the integrand $|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3$.

$$
\begin{aligned}
|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3 &= \left(\sqrt{(b_2\cos\phi_2 - b_1\cos\phi_1)^2 + (b_2\sin\phi_2 - b_1\sin\phi_1)^2 + (L)^2}\right)^3 && (30)\\
&= \Big(b_2^2\cos^2\phi_2 - 2b_1b_2\cos\phi_1\cos\phi_2 + b_1^2\cos^2\phi_1 \\
&\qquad + b_2^2\sin^2\phi_2 - 2b_1b_2\sin\phi_1\sin\phi_2 + b_1^2\sin^2\phi_1 + L^2\Big)^{\frac{3}{2}} && (31)\\
&= \Big(b_2^2\left(\cos^2\phi_2 + \sin^2\phi_2\right) - 2b_1b_2\Big(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\Big) \\
&\qquad + b_1^2\left(\cos^2\phi_1 + \sin^2\phi_1\right) + L^2\Big)^{\frac{3}{2}} && (32)\\
|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3 &= \Big(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\phi_1 - \phi_2)\Big)^{\frac{3}{2}} && (33)
\end{aligned}
$$

Substitute equations 29 and 33 in to equation 9:

$$
\boldsymbol{F}_{21} = \frac{\mu_\circ I_1 I_2 b_1 b_2}{4\pi}\oint_{l_2}\oint_{l_1}\frac{(\boldsymbol{a}_x\cos\phi_2 + \boldsymbol{a}_y\sin\phi_2)\,(b_1 - b_2\cos(\phi_1 - \phi_2)) - \boldsymbol{a}_z L\cos(\phi_1 - \phi_2)}{(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\phi_1 - \phi_2))^{\frac{3}{2}}}d\phi_1 d\phi_2 \tag{34}
$$

Equation 34 will need to be integrated numerically. The geometry of the problem, however, suggests that the x and y components will be zero, because the loops are symmetric with respect to the z-axis. This can be shown mathematically by looking at the x and y components of the force vector:

$$
\begin{aligned}
(\boldsymbol{F}_{21})_{xy} &= \frac{\mu_\circ I_1 I_2 b_1 b_2}{4\pi}\oint_{l_2}\oint_{l_1}(\boldsymbol{a}_x\cos\phi_2 + \boldsymbol{a}_y\sin\phi_2)\,A(\phi_1,\phi_2)d\phi_1 d\phi_2 && (35)\\
&= \frac{\mu_\circ I_1 I_2 b_1 b_2}{4\pi}\oint_{l_2}(\boldsymbol{a}_x\cos\phi_2 + \boldsymbol{a}_y\sin\phi_2)\oint_{l_1}A(\phi_1,\phi_2)d\phi_1 d\phi_2 && (36)\\
&= \frac{\mu_\circ I_1 I_2 b_1 b_2}{4\pi}\oint_{l_2}(\boldsymbol{a}_x\cos\phi_2 + \boldsymbol{a}_y\sin\phi_2)\,B(\phi_2)d\phi_2 && (37)
\end{aligned}
$$

With:

$$
A(\phi_1,\phi_2) = \frac{b_1 - b_2\cos(\phi_1 - \phi_2)}{(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\phi_1 - \phi_2))^{\frac{3}{2}}} \tag{38}
$$

$$
B(\phi_2) = \oint_{l_1}A(\phi_1,\phi_2)d\phi_1 \tag{39}
$$

All that is left is to evaluate equation 39 to get $B(\phi_2)$. There is a degree of freedom in choosing limits of integration, because the integration may start anywhere along the closed path l_1. For reasons that will become apparent, choose ϕ_2 and $\phi_2 + 2\pi$.

$$
B(\phi_2) = \int_{\phi_2}^{\phi_2+2\pi}A(\phi_1,\phi_2)d\phi_1 \tag{40}
$$

Now do a substitution on $A(\phi_1, \phi_2)$ and transform the limits:

$$\alpha = \phi_1 - \phi_2 \tag{41}$$

$$\phi_1 = \alpha + \phi_2 \tag{42}$$

$$d\phi_1 = d\alpha \tag{43}$$

$$\phi_1 = \phi_2 \rightarrow \alpha = 0 \tag{44}$$

$$\phi_1 = \phi_2 + 2\pi \rightarrow \alpha = 2\pi \tag{45}$$

The result is:

$$A = \frac{b_1 - b_2\cos(\alpha)}{\left(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\alpha)\right)^{\frac{3}{2}}} \tag{46}$$

$$B = \int_0^{2\pi} A d\alpha \tag{47}$$

From equations 46 and 47 it is apparent that $B(\phi_2)$ *is constant with respect to* ϕ_2 and can be moved outside the integral in equation 37.

$$(\boldsymbol{F}_{21})_{xy} = \frac{\mu_\circ I_1 I_2 b_1 b_2 B}{4\pi}\int_0^{2\pi}\left(\boldsymbol{a}_x\cos\phi_2 + \boldsymbol{a}_y\sin\phi_2\right)d\phi_2 \tag{48}$$

Knowing that the integrals of sine and cosine $\left(\int_0^{2\pi}\sin x dx = \int_0^{2\pi}\cos x dx = 0\right)$ over one period are zero, $(\boldsymbol{F}_{21})_{xy}$ will be zero. Note that the x and y components of the force at each point on loop 2 are *not* zero, but the total x and y components of the force *are* zero. Equation 34 reduces to:

$$\boldsymbol{F}_{21} = -\frac{\mu_\circ I_1 I_2 b_1 b_2 L}{4\pi}\boldsymbol{a}_z\oint_{l_2}\oint_{l_1}\frac{\cos(\phi_1 - \phi_2)}{\left(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\phi_1 - \phi_2)\right)^{\frac{3}{2}}}d\phi_1 d\phi_2 \tag{49}$$

Results

The magnitude of the force on loop 2 is:

$$F_{21} = |\boldsymbol{F}_{21}| = \frac{\mu_\circ I_1 I_2 b_1 b_2}{4\pi}\Lambda \tag{50}$$

With:

$$\Lambda = L\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(\phi_1 - \phi_2)}{\left(b_1^2 + b_2^2 + L^2 - 2b_1b_2\cos(\phi_1 - \phi_2)\right)^{\frac{3}{2}}}d\phi_1 d\phi_2 \tag{51}$$

Application

Set the force on loop 2 equal to the weight of loop 2. Assume $I_1 = I_2 = N_1 i_1 = N_2 i_2 = Ni$ and $b_1 = b_2 = b$.

$$F_{21} = \frac{\mu_\circ N^2 i^2 b^2}{4\pi}\Lambda = \rho_m A\,(2\pi b N)\,g \tag{52}$$

$$\Lambda = \rho_m A\,(2\pi b N)\,g\frac{4\pi}{\mu_\circ N^2 i^2 b^2} = \frac{8\pi^2 g \rho_m A}{\mu_\circ N i^2 b} \tag{53}$$

$$R = \frac{l}{\sigma A} = \frac{2\pi b N}{\sigma A} \tag{54}$$

$$i^2 = \frac{P_1}{R} = \frac{P_1 \sigma A}{2\pi b N} \tag{55}$$

$$\Lambda = \frac{8\pi^2 g \rho_m A}{\mu_\circ N b}\frac{2\pi b N}{P_1 \sigma A} = \frac{16\pi^3 g \rho_m}{\mu_\circ \sigma P_1} \tag{56}$$